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On the Local Structure of the Phase Separation Line in the Two-Dimensional Ising System[1]

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We investigate the structure of the phase separation line between the pure phases in the two-dimensional Ising model, the liquid and vapor phase in lattice gas language, at low temperatures. The fluctuations in the location of this line are known to diverge in the thermodynamic limit, something which is also believed to happen to the continuum liquid–vapor interface in three dimensions (in the absence of the gravitational field). We show that despite this global divergence it is possible to define precisely the local structure of the phase separation line. This has a finite, exponentially small, width at low temperatures which is related by a central limit theorem[1] to the width of the global fluctuations on the appropriate (divergent) length scale. The latter has been computed explicitly[2] for all temperatures below the critical temperature T_c, where it diverges as $(T_c - T)^{-1/2}$. We also prove a Gibbs formula for the surface tension at low temperature, which relates it to the local structure of the phase separation line.

KEY WORDS: Phase separation; two-dimensional Ising model; surface tension.

1. INTRODUCTION

We investigate the intrinsic structure of the phase separation line in the two-dimensional Ising model. The motivation for this study comes from considering the liquid and vapor phase of a simple fluid coexisting in a

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cubical box V of side L in $\mathbb{R}^3$. In the presence of the gravitational field one observes that the region which separates the two phases, called the interface, has a finite thickness and is stable, i.e., fluctuations of the interface are finite and independent of the size of the box V. However if we remove the gravitational field there are arguments (see, e.g., Refs. 3 and 4) showing that the mean square fluctuation of the height of the interface in the middle of the box diverges as $\log L$ when L tends to infinity. This divergence is due to fluctuations at very long wavelength; see Ref. 4 for an analysis in terms of Goldstone modes corresponding to the breaking of a continuous (translational) symmetry. We would therefore expect that despite this divergence there would be, as for example in the two-dimensional harmonic crystal which shows a similar divergence, a well-defined local intrinsic structure of the interface which is independent of the gravitational field. Indeed it is this intrinsic structure which is studied in the classical theories of van der Waals,[5] and Cahn and Hilliard.[6] These theories predict a divergence of the interface width, behaving like the bulk correlation length when $T \to T_c$; see the reviews of Widom,[7] Weeks,[8] Rowlinson,[9] and literature cited there.

It would clearly be very desirable to prove rigorous results for the continuum liquid–vapor interface. This seems very difficult, however, and even the precise microscopic definition of the interface is nontrivial. We can, however, study in detail the analogous situation for the two-dimensional Ising model at low temperature where Gallavotti[1] showed that the mean square fluctuation of the height of the interface, or phase separation line, diverges as $\sqrt{L}$ in the thermodynamic limit. We find indeed that despite this divergence of the location of the phase separation line it is possible to define an intrinsic thickness which has a limit as $L \to \infty$ and is finite (exponentially small) at low temperatures. Unfortunately our methods, based on the work of Gallavotti, are limited to low temperatures and we cannot, therefore, make any precise statement about the divergence of this intrinsic thickness as $T \to T_c$ from below. Our results however do suggest that the thickness may behave, for all $T < T_c$, as the fluctuations in the position of the phase separation line divided by $\sqrt{L}$. The latter has been computed explicitly by Abraham and Reed[2] by studying the behavior of the magnetization as a function of height on the scale $\sqrt{L}$. They find that this diverges as $(T_c - T)^{-1/2}$, which is different from the divergence of the correlation length which behaves as $(T_c - T)^{-1}$. We shall discuss this more later: we now give some background.

2. BACKGROUND

As is well known the Ising model, in two or more dimensions, can, at low temperatures, exist in two distinct pure phases. These phases are characterized by a spontaneous magnetization $\pm m^*(T)$; $m^*(T) > 0$ for $T < T_c$. (See, e.g., Ref. 10 for a review.) In three or more dimensions Dobrushin[11] proved that one can by a suitable choice of the boundary conditions, one favoring the "+" state in the top half of the box and the "−" state in the bottom half, construct at low temperatures a state of the infinite system such that as $z \to +\infty\,(-\infty)$ the state looks like the positively (negatively) magnetized pure state while around $z = 0$ the magnetization $m(z)$ changes from $-m^*$ to $+m^*$, i.e., exactly the kind of state we said was impossible in the continuum system due to the adding up of arbitrary small deformations possible in the latter. A weaker version of this result, valid, however, over a wider temperature range, was later proved simply by van Beijeren.[12] Given the existence of such an infinite volume state, it is easy to define what is meant by an interface or phase separation surface, e.g., we can define the interface as the region of space where $[m^* - m(z)] > \epsilon m^*$ for some $\epsilon > 0$. Of course there is some arbitrariness in this definition but any other reasonable definition will give similar results for quantities such as the width of the interface as a function of the temperature.

It is also possible in this case to relate the surface tension τ to the interface width: taking the derivative of τ with respect to β[14] gives

$$\frac{d\tau}{d\beta} = \sum_{z \in \mathbb{Z}} \left(\langle e_z \rangle^+ - \langle e_z \rangle^\pm\right) \tag{2.1}$$

where e_z denotes the local energy density and the sum is over a vertical line. $\langle\ \rangle^+$ is the expectation value in the pure state with positive spontaneous magnetization and $\langle\ \rangle^\pm$ is the expectation value in the state constructed by Dobrushin. Since $|\langle e_z \rangle^+ - \langle e_z \rangle^\pm|$ is very different from zero only in the vicinity of the interface Eq. (2.1) provides a link between the structure of the latter and τ. Equation (2.1) can be proven to hold at low temperatures for $d \geqslant 3$; it is presumably valid, however, all the way to the roughening temperature.

The situation is, however, very different in the two-dimensional Ising model. If we construct the same state $\langle\ \rangle^\pm$ as Dobrushin did in three dimensions then, as already noted, we find[1] that the mean square fluctuation of the height of the phase separation line in the middle of the box V diverges as $\sqrt{L}$ when L becomes infinite. The profile of the magnetization is thus washed out in the thermodynamic limit and the resulting state is translation invariant with $\langle e_z \rangle^+ = \langle e_z \rangle^\pm$ for any fixed z, the phase separation line having "disappeared to infinity" as $\pm\sqrt{L}$. To "capture" this phase separation line, Abraham and Reed[2] computed the scaled magnetization profile for this model,

$$m(\alpha) = \lim_{L \to \infty} \langle \sigma_z \rangle_L^\pm, \qquad \text{with } z = \alpha L^\delta$$

where $\langle\ \rangle_L^{\pm}$ is the state with $\pm$ boundary conditions in the box V with cross section $2L$. They found $m(\alpha) = 0$ if $\delta < 1/2$ and $m(\alpha) = m^* \operatorname{sgn} \alpha$ if $\delta > 1/2$. For $\delta = 1/2$ they found a Gaussian profile. We will show in Section 4 that at low temperatures all local averages have a similar Gaussian profile due to a central limit theorem proved in the work of Gallavotti.(1)

The rescaled magnetization profile computed by Abraham and Reed does not *a priori* provide an intrinsic definition of the thickness of the phase separation line. Rather it is tempting to say that in the two-dimensional Ising model the width of the phase separation line is infinite. However, as we shall show the local structure of the phase separation line is not destroyed in the thermodynamic limit, although the global fluctuations lead to the divergence of its position. In particular we shall show that it is possible to define an intrinsic width which is finite at low temperature. While we cannot see this local structure on the scale of $\sqrt{L}$ we can use the results of Abraham and Reed to get some information on the intrinsic width at low temperature and this presumably will remain valid up to T_c. We shall also derive a formula for $d\tau/d\beta$ similar to (2.1) providing a relation between the intrinsic phase separation line and the surface tension in this system.

The outline of the rest of the paper is as follows. We start Section 3 by describing some properties of the pure phase [part (a)]. We then specify the boundary conditions which lead to the coexistence of the two phases and define the separation line between them [part (b)]. In part (c) we study more precisely the statistical distribution of the separation line and decompose it into elementary constituents. In (d) we state the main result of Ref. 1 on the global fluctuation of the phase separation line. Finally in (e) we give a precise definition of what we mean by the local structure of the phase separation line. In Section 4 we make the connection between this local structure and the results of Abraham and Reed(2) on the scaled magnetization profile. Section 5 proves the formula for the surface tension, while Section 6 describes some speculations about the behavior of the local structure near T_c. Appendix A contains a proof of the exponential approach to the limit in the pure phases for correlations far from the boundary. Appendix B proves the equality of two different definitions of correlation functions of deformations of the phase separation line.

3. THE PHASE SEPARATION LINE

(a) We consider the two-dimensional Ising model on the square lattice, $\mathbb{L} = \{(i_1, i_2) = i : (i_1, i_2 + 1/2) \in \mathbb{Z}^2\}$ with nearest-neighbor interactions of the form $-J\sigma_i\sigma_j$, $\sigma_i = \pm 1$, $J > 0$. There is no external magnetic field. Below a certain temperature $T_c = \beta_c^{-1}$ there exist two pure phases $\langle \cdot \rangle^+$, respectively, $\langle \cdot \rangle^-$, which are characterized by a positive, respectively negative, spontaneous magnetization $\langle \sigma_i \rangle^+ = -\langle \sigma_i \rangle^- = m^* > 0$. These two states are obtained using the + or − boundary conditions, i.e., by considering a system in a finite box V with the spins outside V fixed at the value $+1$ (-1) and then taking the thermodynamic limit $V \to \mathbb{L}$. The limit for the free energy and the correlation functions exists and for the latter the convergence is exponentially fast:

Proposition 3.1. For any $\beta \neq \beta_c$, there exist two constants $a > 0$, $K < \infty$ such that for all $V_1, V_2 \subset \mathbb{L}$, and $A \subset V_1 \cap V_2$

$$|\langle \sigma_A \rangle_{V_1}^+ - \langle \sigma_A \rangle_{V_2}^+| \leqslant K \sum_{i \in A} \sum_{j \in V_1 \Delta V_2} \exp(-a|i-j|) \tag{3.1}$$

where $\sigma_A = \prod_{i \in A} \sigma_i$. $\langle \; \rangle_V^+$ is the state in V with + boundary conditions and $V_1 \Delta V_2 = (V_1 \setminus V_2) \cup (V_2 \setminus V_1)$. Moreover, a may be taken proportional to β as $\beta \to \infty$.

The proof of this lemma can be found in Appendix A.

(b) We study now a different boundary condition, one suitable to describe the coexistence of the two pure phases.[1,11,12] Let $A_{L,M}$ be the box $\{(i_1, i_2) \in \mathbb{L} : |i_1| \leqslant L, |i_2| \leqslant M - 1/2\}$. We choose the boundary condition by fixing the value of σ_i for $i \notin A_{L,M}$ as follows:

$$\sigma_i = +1 \qquad \text{if } i_2 > 0$$
$$\sigma_i = -1 \qquad \text{if } i_2 < 0$$

The corresponding Gibbs state is denoted by $\langle \cdot \rangle_{L,M}^{\pm}$.

It is convenient to give a geometrical description of the configurations of the system (see, e.g., Refs. 1 and 10): we draw the unit segment on the dual lattice $\mathbb{L}^* = \{(i_1, i_2) : (i_1 + 1/2, i_2) \in \mathbb{Z}^2\}$ between each pair of neighboring spins with opposite signs. Two segments are *adjacent* if they touch each other. A set of segments is *connected* if any two segments of this set can be joined by a path of adjacent segments. We call the connected sets of segments *contours*. Owing to the choice of the boundary condition it is easy to see that all contours are closed except for one which starts with the segment between $(-(L+1), 1/2)$ and $(-(L+1), -1/2)$ where we have, respectively, a spin $+1$ and -1, and which ends with the segment between $(L+1, 1/2)$ and $(L+1, -1/2)$. We denote this special contour by λ and the others by $\gamma_1, \ldots, \gamma_n$.

Let $|\gamma_i|$ and $|\lambda|$ be the lengths of the contours (i.e., the number of segments). Then we can express the energy of a configuration, up to a constant $C_{L,M}$ independent of the configuration as

$$2J\left(\sum_{i=1}^{n} |\gamma_i| + |\lambda|\right)$$

We consider the set of all configurations in $A_{L,M}$ with a given common λ. Since by definition two different contours are disconnected, all spins located in the centers of the cells of $\mathbb{L}^*$ which touch λ (by a segment or a corner) have well-defined values prescribed by λ. Let $A_{L,M}(\lambda)$ be the complementary set in $A_{L,M}$. $A_{L,M}(\lambda)$ is composed of several components. We have $A^+_{L,M}(\lambda)$, respectively, $A^-_{L,M}(\lambda)$, which are connected to the top, respectively, to the bottom of $A_{L,M}$. The boundary conditions (b.c.) for $A^+_{L,M}(\lambda)$ and $A^-_{L,M}(\lambda)$ are the + one and the − one, respectively (see Fig. 1a). Other components in $A_{L,M}(\lambda)$ are completely surrounded by λ. If we compute the expectation value of a spin at site $i \in A^+_{L,M}$ under the condition that λ is fixed then, by (3.1) the result is, uniformly in L and M, exponentially close to $\langle \sigma_i \rangle^+_{L,M}$ with the distance between i and λ. We call λ the *phase separation line*.

Remark. We note that if we change slightly the definition of the contours, as is done sometimes, so that a contour does not intersect itself, then λ is also changed. However the λ obtained in this way has also the property that it defines two regions where we find the pure phases. This is

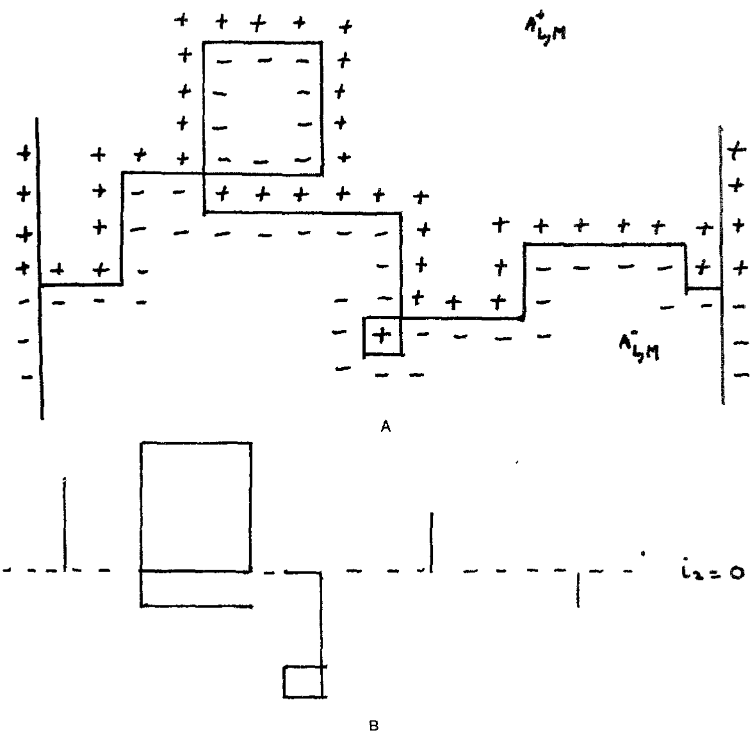

Fig. 1. (A) Phase separation line λ with the boundary conditions specified by λ. (B) The deformations associated to the line λ with origins on $i_2 = 0$.

the property of λ we use, and, at least at low temperatures, the freedom in the definition of λ does not play any role.

Summing over all configurations with a given λ we obtain the *probability* $P_{L,M}(\lambda)$ of λ in $A_{L,M}$:

$$P_{L,M}(\lambda) = \frac{e^{-2\beta J|\lambda|} Z_{L,M}(\lambda)}{Z_{L,M}^{\pm}} \tag{3.2}$$

where $Z_{L,M}^{\pm}$ is the partition function in $A_{L,M}$ with $\pm$ b.c. and $Z_{L,M}(\lambda)$ is the partition function of the system in $A_{L,M}(\lambda)$ with the boundary conditions specified by λ. $Z_{L,M}(\lambda)$ is a product of partition functions with pure $+$ boundary conditions (or $-$ b.c. which is equivalent by symmetry) in the different connected components of $A_{L,M}(\lambda)$. Dividing both the numerator and the denominator of (3.2) by $Z_{L,M}^{+}$, the partition function in $A_{L,M}$ with $+$ b.c. we may take the limit $M \to \infty$ since the following limit exists (notice that, for L fixed, we have a one-dimensional system):

$$\lim_{M \to \infty} \frac{Z_{L,M}^{\pm}}{Z_{L,M}^{+}} = \bar{Z}_L$$

and

$$\lim_{M \to \infty} \frac{Z_{L,M}(\lambda)}{Z_{L,M}^{+}} = \exp\left[-\bar{U}_L(\lambda) \right]$$

Furthermore, for β large enough[1,11] $|\bar{U}_L(\lambda)| \leqslant K|\lambda|$ for some constant K, independent of L with $K \to 0$ as $\beta \to \infty$. We get therefore the probability distribution $P_L(\lambda)$ of λ for the system confined in $A_L = \{ i \in \mathbb{L}, |i_1| \leqslant L \}$ in the form

$$P_L(\lambda) = \frac{\exp\left[-2\beta J|\lambda| - \bar{U}_L(\lambda) \right]}{\bar{Z}_L}$$

(c) We study now the probability distribution $P_L(\lambda)$ defined on the set of all phase separation lines in A_L. This set is the configuration space for $P_L(\lambda)$. To do this it is useful to have another description of this configuration space. We define for all points $(x, 0)$ the width $E_x(\lambda)$ of λ at the point $(x, 0)$ as the distance between the highest and lowest intersection points of λ with the vertical line $i_1 = x$. We wish to describe λ by its difference from the case $T = 0$ when λ is a straight line and $E_x(\lambda) = 0$ for all x. Therefore we decompose λ into connected components of two kinds: the *regular components* and the *deformations*. The regular components are horizontal pieces of λ where the thickness of λ is zero. The deformations are the parts of λ where the thickness of λ is nonzero. They are connected to regular components because λ is a connected line (see Fig. 1b).

We measure the size of a deformation W by the quantity

$$\pi(W) = |W| - |j_1 - i_1|$$

where $|W|$ is the number of segments in W and (i_1, i_2) and (j_1, j_2) are the coordinates of the first and last points of W, respectively. The first coordinate of the first point of W is called the *origin* of W. For each point i of the horizontal line $i_2 = 0$, we define

$$D_i(\lambda) = \begin{cases} (j_2 - i_2) & \text{if there is a deformation } W \text{ in } \lambda \\ & \text{with origin } (i_1, i_2) \text{ with } i_1 = i \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

Notice that if the only deformations were vertical pieces then the variables D_i would define λ uniquely. At very low temperature this is essentially the case. Describing λ from the left to the right we find first a regular component (see Fig. 1b) then some deformation W. After W we have again a regular component at height j_2, where (j_1, j_2) denotes the last point of W, and so on.

By construction two different deformations are separated by at least one regular component. Therefore their projections on the line $i_2 = 0$ are disconnected. To every λ we can associate a set of deformations and since the relative vertical position of a deformation is fixed by the height of the regular component at its left-hand side we need to specify only the origins of the deformations. We write $\theta = \theta(\lambda) = (W_{i_1}, \ldots, W_{i_n})$ if λ has the deformations $W_{i_1}, \ldots, W_{i_n}$ with origins $i_1, \ldots, i_n$. Since the first and the last regular components of λ are at the same height we must have $\sum_i D_i(\lambda) = 0$. We say that $\theta = (W_{i_1}, \ldots, W_{i_k})$ is *admissible* if the projections of $W_{i_1}, \ldots, W_{i_k}$ on the line $i_2 = 0$ are pairwise disconnected. We have a one-to-one correspondence between phase separation lines λ and admissible θ such that $\sum_i D_i(\theta) \equiv \sum D_i(\lambda(\theta)) = 0$. [We may write $\theta = \theta(\lambda)$ and $\lambda = \lambda(\theta)$.]

We use this correspondence to define a probability distribution on the set Ω_L of admissible θ by putting

$$P_L(\theta) = \begin{cases} P_L(\lambda(\theta)) & \text{if } \sum_i D_i(\lambda(\theta)) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since at $T = 0$ the phase separation line is just a straight line λ_0 without any deformation it is convenient to introduce $U_L(\theta) = \overline{U}_L(\lambda(\theta)) - \overline{U}_L(\lambda_0)$. This function is the effective energy between the deformations. Since $\sum_{k=1}^n \pi(W_{i_k}) + 2L = |\lambda|$ we have

$$P_L(\theta) = Z_L^{-1} \exp\left[-2\beta J \sum_{k=1}^{n} \pi(W_{i_k}) - U_L(\theta) \right] \tag{3.4}$$

where Z_L is the new normalization constant.

The description which we have now obtained for λ is similar to the description of a configuration of the pure phase in terms of contours[13] and the same techniques can be used to study λ. The important difference is that the energy of θ, $U_L(\theta)$, is given by many-body long-ranged interactions with exponential decay. The factors $\exp[-2\beta J\pi(W_{i_k})]$ play the role of small activities for the deformations W_{i_k}. Let θ be any finite admissible set of deformations. We define the correlation function $\rho_L(\theta)$ by

$$\rho_L(\theta) = \sum_{\theta \subseteq \theta'} P_L(\theta') \tag{3.5}$$

The study of λ is done via the study of the correlation functions, e.g., one can use a technique related to the Kirkwood–Salsburg equations.[1, 13]

The following results, for large β, follow from Ref. 1 or can be proven as in Ref. 11:

Proposition 3.2. There exist two constants c and α such that, for β large,

(a)
$$\rho_L(\theta \leqslant \exp[-\alpha\beta\pi(\theta)]$$

where $\pi(\theta) = \sum \pi(W)$, the sum being over all deformations W of θ, and

(b)
$$P_L(E(x) \geqslant N) \leqslant P_L(\pi(W) \geqslant N) \leqslant c \exp(-\alpha\beta N)$$

in particular,

$$P_L(E(x)) < \infty$$

where $P_L(E(x))$ denotes the expectation value of $E(x)$ with respect to P_L.

Proof. (a) follows from the estimates of Section 6 in Ref. 1 provided one identifies the correlation functions here and those of Ref. 1. This is done in Appendix B. (b) follows from (a) and the fact that if $E(x) > 0$ then there exists a deformation W whose projection on the line $i_2 = 0$ contains the point $(x, 0)$. Clearly $E(x) \leqslant \pi(W)$, and the number of deformations with given $\pi(W)$ and given origin is bounded by $C^{\pi(W)}$ for some constant C since the deformations are connected subsets of the phase separation line. ■

Remark. Using (3.5) it is easy to estimate the probability that there is a very large deformation in λ; [see Ref. 1, Eqs. (8.18) and (8.19)]. Let $F(L)$ be some positive increasing function of L. Since in A_L we can have at most $2(L+1)$ deformations in a phase separation line λ, the probability that there exists one deformation with $\pi(W) \geqslant F(L)$ is smaller than

$$2(L+1)c \exp\left[-\alpha\beta F(L)\right] \tag{3.6}$$

Thus the probability that there is one deformation W with $\pi(W) \geqslant \log L$ tends, for large β, to zero when L tends to infinity.

(d) The main result of Gallavotti[1] is the computation of the distribution of the two random variables h^+ and h^- defined, respectively, as the highest and lowest intersection points of λ with the vertical line $i_1 = 0$.

Proposition 3.3. We have

$$\lim_{L\to\infty} P_L\left(h^+ \leqslant \frac{\alpha(L)^{1/2}}{2}\right) = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\alpha} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt \tag{3.7}$$

and

$$\lim_{L\to\infty} P_L\left(h^- \leqslant \frac{\alpha(L)^{1/2}}{2}\right) = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\alpha} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt \tag{3.8}$$

with $0 < \sigma^2 < \infty$, a constant which is temperature dependent and is given by (8.8) in Ref. 1.

Remarks. (1) Notice that we have the same limiting distribution for the two random variables h^+ and h^- since we have a scaling by $(L)^{1/2}$ and the thickness of λ is bounded by $\log L$ as $L \to \infty$ (see preceding remark). (2) If we study only the stochastic process D_i we get the same distribution for $\sum_{i=-(L+1)}^{0} D_i$:

$$\lim_{L\to\infty} P_L\left(\sum_{i=-(L+1)}^{0} D_i \leqslant \frac{\alpha(L)^{1/2}}{2}\right) = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\alpha} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt \tag{3.9}$$

(3.9) expresses the fact that, although the random variables D_i are not independent, they nevertheless satisfy a central limit theorem.

(e) We give now a description of the local structure of the phase separation line λ in the thermodynamic limit ($L \to \infty$). While λ itself is at infinity in this limit the correspondence between λ and admissible sets of deformations allows us to study the local structure of λ using the deformations. An analogous situation exists for a one-dimensional harmonic chain. In the thermodynamic limit only the correlation functions of the difference variables are well defined. However, we may reconstruct from them a probability distribution on configurations of the variables themselves by fixing, e.g., the variable at the origin equal to zero.

Let Ω be the configuration space of all admissible sets of deformations. On this space we shall consider the probability distribution specified by the correlation functions

$$\lim_{L\to\infty} \rho_L(\theta) = \rho(\theta) \tag{3.10}$$

which are the limits of the $\rho_L(\theta)$. The construction envisioned here is the same as the construction of Gibbs states for an infinite system via Kolmogorov's theorem: the correlation functions determine a compatible set of probabilities for cylindrical events and this defines a probability P on Ω which is translation invariant in the i_1 direction.

An important property of this probability distribution is that the correlation functions approach their limit exponentially fast in L and are exponentially clustering, at least for β large. The following proposition can be proven, using the technique of Ref. 1, in a way similar to the proof of Proposition 3.2.

Proposition 3.4. There exist constants $\kappa > 0$ and $C < \infty$, $K < \infty$ such that, for β large enough, we have the following.

(a) For all L, L' with $L' > L$,

$$\sum_{\substack{\theta \subset A_L \\ \theta = (W_t,\, t \in D)}} |\rho_L(\theta) - \rho_{L'}(\theta)| \leqslant C \sum_{\substack{t \in D \\ s\,:\, L' - L \leqslant |s| \leqslant L'}} \exp[-\kappa\beta|s-t|] \quad (3.11)$$

where D is a finite subset of the line $i_2 = 0$ and $|s|$ runs on this line between $L' - L$ and L'.

(b) Let θ_1 and θ_2 be two admissible finite sets of deformations such that the deformations of θ_1 are at the left-hand side of each deformation of θ_2. Let $d(\theta_1, \theta_2)$ be the distance between the projections of θ_1 and those of θ_2 on the line $i_2 = 0$. Then

$$|\rho(\theta_1 \cup \theta_2) - \rho(\theta_1)\rho(\theta_2)| \leqslant K \exp[-\kappa\beta d(\theta_1, \theta_2)] \quad \blacksquare \quad (3.12)$$

Using the deformations we are able to study the local structure for the phase separation line in the thermodynamic limit. In analogy with the one-dimensional harmonic chain mentioned above, one can reconstruct a phase separation line λ from a given admissible set of deformations by specifying that the origin of the first deformation to the right of $(0, 0)$ is at height $i_2 = 0$. This construction allows us to define a probability distribution P on the set of interfaces satisfying this condition, which is induced by the probability distribution on Ω constructed above. As the previous proposition shows, the probability of having a given shape in the interval $[-N, +N]$ is almost independent of the shape of λ far from this interval. So the effect of the distant deformations is relevant only for locating this piece of λ vertically but does not influence the "intrinsic shape" of λ.

To summarize: We have shown that at low temperature there is a well-defined local structure for the phase separation line λ even in the thermodynamic limit.

A measure of the *intrinsic width* of the interface is $\langle D_0^2 \rangle^{1/2}$ where $\langle\ \rangle$ is the expectation value in the state P just defined and D_0 is given by (3.3).

By Proposition 3.2 this width is finite at low temperature. Note that by Proposition 3.1 the finiteness of this width implies that the width of the region where $m^* - |\sigma_i^{\pm}(\lambda)|$ is larger than ϵm^*, $\sigma_i^{\pm}(\lambda)$ being the expectation of σ_i given the line λ, has a finite expectation value. This is another way of saying that the intrinsic width is finite at low temperatures.

4. THE MAGNETIZATION PROFILE

Using the ideas of Section 3, we shall now compute a rescaled magnetization profile and compare our result with the explicit computation of Abraham and Reed.[2]

Let A be any finite set of $\mathbb{L}$ such that $(0, 1/2) \in A$. Let $\sigma_A = \prod_{i \in A} \sigma_i$ and $A_t = \{(i_1, i_2) \in \mathbb{L}; (i_1, i_2 - t) \in A\}$. We compute the profile of this observable on the scale $(L)^{1/2}$, when L goes to infinity.

Proposition 4.1. Let $t = \alpha(L/2)^{1/2}$, then for large β

$$\text{(a)} \quad \lim_{L \to \infty} P_L(\sigma_{A_t}) = \langle \sigma_A \rangle^+ \text{ if } |A| \text{ is even} \tag{4.1}$$

$$\text{(b)} \quad \lim_{L \to \infty} P_L(\sigma_{A_t}) = \operatorname{sgn} \alpha \langle \sigma_A \rangle^+ \varphi\left(\frac{|\alpha|}{\sqrt{2}\,\sigma} \right), \quad \text{if } |A| \text{ is odd}$$

$$\varphi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{4.2}$$

Proof. We consider the special case where $A = \{(0, 1/2)\}$ and put $\sigma_t \equiv \sigma_{A_t}$. Let $\epsilon > 0$. We compute first the conditional expectation value of σ_t given that $h^+(\lambda) \leqslant (\alpha - \epsilon)\sqrt{L}/\sqrt{2}$. From (3.6) we have that the distance from t to λ tends to infinity (with probability 1) when $L \to \infty$. Since $t \in A_L^+(\lambda)$ (see definition of the interface) this conditional expectation value is by Proposition 3.1 $\langle \sigma_0 \rangle^+$ at the limit $L \to \infty$. Similarly the conditional expectation value of σ_t given that $h^-(\lambda) \geqslant (\alpha - \epsilon)\sqrt{L}/\sqrt{2}$ is $\langle \sigma_0 \rangle^- = -\langle \sigma_0 \rangle^+$ at the limit $L \to \infty$. Taking ϵ arbitrarily small and using (3.7), (3.8) we get

$$\lim_{L \to \infty} P_L(\sigma_t) = \langle \sigma_0 \rangle^+ \frac{1}{(2\pi)^{1/2}\sigma} \left[\int_{-\infty}^{\alpha} \exp\left(-\frac{u^2}{2\sigma^2} \right) du - \int_{\alpha}^{\infty} \exp\left(-\frac{u^2}{2\sigma^2} \right) du \right]$$

$$= \langle \sigma_0 \rangle^+ \operatorname{sgn} \alpha \varphi\left(\frac{|\alpha|}{\sqrt{2} \cdot \sigma} \right)$$

The proof for the general case is just the same. ■

Abraham and Reed computed the magnetization profile and found for all $T < T_c$, the critical temperature, that

$$\lim_{L\to\infty} P_L(\sigma_t) = \langle\sigma_0\rangle^+ \operatorname{sgn}\alpha\varphi\left(\frac{b|\alpha|}{\sqrt{2}}\right) \tag{4.3}$$

with

$$b^2 = \sinh 2(K - K^*), \qquad e^{-2K^*} = \tanh K, \qquad K = \beta J \tag{4.4}$$

Therefore we have that the σ defined by (3.7) is given by

$$1/\sigma^2 = b^2 = \sinh 2(K - K^*) \tag{4.5}$$

The link between (3.7) and (3.8) proven for β large on the basis of Gallavotti's result and Eq. (4.3) of Abraham and Reed is straightforward: in both cases the quantity computed is the same, namely, the distribution of the middle part of λ on the scale $\sqrt{L}$. This explains the Gaussian character of the rescaled magnetization profile. The result (4.3) suggests very strongly that (3.7), (3.8), and (3.9) are valid up to T_c with σ^2 given by (4.5).

Remarks. (1) It is not difficult to show that in the limit $L\to\infty$

$$\sigma^2 = \sum_i \langle D_0 D_i\rangle \tag{4.6}$$

as expected from (3.9). (2) Notice that if σ^2 is given by (4.5) then

$$\sigma^2 \sim (T_c - T)^{-1}, \qquad T \sim T_c \tag{4.7}$$

We shall return to this point in Section 6.

5. THE SURFACE TENSION

In this section we prove a formula for the derivative of the surface tension τ with respect to β which is similar to the Gibbs formula proven in Ref. 14. This gives some connection between τ and the phase separation line λ. We need more notation. Let

$$\tau_{L,M} = \log(Z^{\pm}_{L,M}/Z^{+}_{L,M}) \tag{5.1}$$

and

$$e_x = -\frac{1}{2}\sigma_x J \sum_{\substack{y:\\ \|x-y\|=1}} \sigma_y \tag{5.2}$$

the local energy density at $x \in \mathbb{L}$. We define the surface tension by

$$\tau = -\lim_{L\to\infty} \frac{1}{2(L+1)} \lim_{M\to\infty} \tau_{L,M} \tag{5.3}$$

and refer the reader to Ref. 15 for a proof of the equivalence of this definition to other definitions. We compute first

$$-\frac{d\tau_{L,M}}{d\beta} = \sum_{x \in A_{L,M}} \left(\langle e_x \rangle^{+}_{L,M} - \langle e_x \rangle^{\pm}_{L,M} \right)$$
$$= \sum_{\lambda \in A_{L,M}} P_{L,M}(\lambda) \sum_{x \in A_{L,M}} \left[\langle e_x \rangle^{+}_{L,M} - \langle e_x \rangle^{\pm}_{L,M}(\lambda) \right]$$

where $\langle \cdot \rangle^{\pm}_{L,M}(\lambda)$ is the expectation value when we fix the phase separation line. We introduce

$$f_{L,M}(i_1 \mid \lambda) = \sum_{i_2} \left[\langle e_{(i_1,i_2)} \rangle^{+}_{L,M} - \langle e_{(i_1,i_2)} \rangle^{\pm}_{L,M}(\lambda) \right] \tag{5.4}$$

so that

$$-\frac{d\tau_{L,M}}{d\beta} = \sum_{i_1} \sum_{\lambda} P_{L,M}(\lambda) f_{L,M}(i_1 \mid \lambda) = \sum_{i_1} P_{L,M}(f_{L,M}(i_1 \mid \cdot)) \tag{5.5}$$

Proposition 5.1. Let β be large enough and let

$$f(0 \mid \lambda) = \sum_{i_2 \in \mathbb{Z}} \left[\langle e_{(0,i_2)} \rangle^{+} - \langle e_{(0,i_2)} \rangle^{\pm}(\lambda) \right]$$

where λ is the phase separation line reconstructed by specifying that the origin of the first deformation at the right of $(0,0)$ is at height $i_2 = 0$. $\langle\ \rangle^{\pm}(\lambda)$ is the state in $A(\lambda)$ [whose definition is analogous to that of $A_{L,M}(\lambda)$] with the boundary conditions specified by λ. Then, with P defined in Section 3(e),

$$-\frac{d\tau}{d\beta} = P(f(0 \mid\ )) \tag{5.6}$$

$P(f(0 \mid\ ))$ being the expected value of $f(0 \mid\ )$ with respect to P.

Proof. We consider first the limit $M \to \infty$ in (5.5) and show that this limit is

$$-\frac{d}{d\beta} \lim_{M \to \infty} \tau_{L,M} = \sum_{i_1} P_L(f_L(i_1(\cdot))) \tag{5.7}$$

with

$$f_L(i_1 \mid \lambda) = \sum_{i_2} \left[\langle e_{(i_1,i_2)} \rangle^{+}_{L} - \langle e_{(i_1,i_2)} \rangle^{\pm}_{L}(\lambda) \right]$$

Since the sum over i_1 is finite we look only at a particular value $i_1 = 0$. We divide the sets of phase separation lines in $A_{L,M}$ respectively in A_L into two classes: $E = \{\lambda : \pi(\theta(\lambda)) \leqslant L^*\}$ and E^c which is the complement of E in $A_{L,M}$ respectively in A_L. (We take M large enough so that E is the same in

A_L and $A_{L,M}$.) If $\lambda \in E^c$ then $|f_{L,M}(0|\lambda)| \leqslant c\pi(\theta(\lambda))$ and $|f_L(0|\lambda)| \leqslant c\pi(\theta(\lambda))$ with some constant c. On the other hand, since $P_{L,M}(\lambda)$, respectively, $P_L(\lambda)$, is exponentially small in $\pi(\theta(\lambda))$ the error $\epsilon(L^*)$ which we make when we replace the set of all interfaces by the set E in computing the expectation values of $f_{L,M}$, respectively, f_L, goes to zero when $L^* \to \infty$. We choose now $M > L^* + 2L'$ with L' large. Since for $\lambda \in E$ and $|i_2| > L^* + L'$ the distance from i_2 to λ is larger than L' we have

$$\left| \sum_{|i_2| > L^* + L'} \left[\langle e_{(0,i_2)} \rangle^+_{L,M} - \langle e_{(0,i_2)} \rangle^\pm_{L,M}(\lambda) \right] \right| \leqslant \epsilon(L')$$

with $\epsilon(L') \to 0$ when $L' \to \infty$ (see Proposition 3.1). The same is true for the corresponding expression in A_L. Again by Proposition 3.1 we have that

$$\begin{aligned} &\left| \sum_{|i_2| \leqslant L^* + L'} \left[\langle e_{(0,i_2)} \rangle^+_L - \langle e_{(0,i_2)} \rangle^\pm_L(\lambda) \right] \right. \\ &\qquad \left. - \sum_{|i_2| \leqslant L^* + L'} \left[\langle e_{(0,i_2)} \rangle^+_{L,M} - \langle e_{(0,i_2)} \rangle^\pm_{L,M}(\lambda) \right] \right| \\ &\qquad \leqslant \sum_{|i_2| \leqslant L^* + L'} \left| \langle e_{(0,i_2)} \rangle^+_L - \langle e_{(0,i_2)} \rangle^+_{L,M} \right| \\ &\qquad + \sum_{|i_2| \leqslant L^* + L'} \left| \langle e_{(0,i_2)} \rangle^\pm_L(\lambda) - \langle e_{(0,i_2)} \rangle^\pm_{L,M}(\lambda) \right| \leqslant \epsilon(L') \end{aligned}$$

We have to prove now that

$$\lim_{L\to\infty} \frac{1}{2(L+1)} \sum_{i_1} P_L(f_L(i_1|\cdot)) = P(f(0|\cdot)) \tag{5.8}$$

First of all we have a uniform bound on $P_L(f_L(i_1(\cdot))$ with respect to i_1 and L. This follows from (3.1), (3.12), and Proposition 3.2 since the influence of a deformation on $f_L(i_1|\lambda)$ decays exponentially with the distance of its origin to i_1 [(3.1) and (3.12)]. Furthermore the contributions of large deformations are exponentially small (Proposition 3.2). Therefore,

$$\lim_{L\to\infty} \frac{1}{2(L+1)} \sum_{L-L^\delta \leqslant |i_1| \leqslant L} P_L(f_L(i_1|\cdot)) = 0 \tag{5.9}$$

if $0 < \delta < 1$. It is therefore sufficient to prove that $P_L(f_L(i_1|\cdot))$ tends to $P(f(0|\cdot))$ exponentially fast with L becoming infinite for $|i_1| \leqslant L - L^\delta$.

It is clear from the definition of $f_L(i|\lambda)$ and $f(i|\lambda)$ and Proposition 3.1 that for $|i_1| \leqslant L - L^\delta$, f_L and f are exponentially close, in N, to $\tilde{f}_N$, where in $\tilde{f}_N$ we replace the separation lines λ by $\tilde{\lambda}_N$ which has the same deformation as λ for those with origin in $[i_1 - N, i_1 + N]$ and none outside this interval. This is true at least for those λ which do not have a too large deformation with its origin outside $[i_1 - N, i_1 + N]$ that would come over the interval $[i_1 - N/2, i_1 + N/2]$. However the probability of such large deformations is, by Proposition 3.2, also exponentially small with N. On the other hand, $\lim_{L\to\infty} P_L(\tilde{\lambda}_N) = P(\tilde{\lambda}_N)$ because to specify $\tilde{\lambda}_N$ is nothing else but to specify a set of deformations with origin on $[-N, +N]$. Combining these two facts and (5.9) the result follows. ■

6. DISCUSSION

We have obtained rigorous results about the local structure of the interface in two dimensions at low temperature. In particular we have shown that the interface region has a finite intrinsic width as it does in three dimensions although this region is at infinity in the two-dimensional lattice. We now make some speculations about the width at higher temperatures in three and two dimensions.

(i) The formula giving the surface tension shows that τ depends only on the local structure of the interface as is the case in the three-dimensional Ising model at low temperatures, e.g., (2.1), when the interface is stable. It is believed[16] that there exists in three dimensions a roughening temperature T_R, which is less than T_c, above which the state constructed by Dobrushin is translation invariant. It is possible that for $T_R \leqslant T < T_c$, $d\tau/d\beta$ will be given by a formula like (5.6). Thus if the local structure of the interface is not affected as we cross T_R the surface tension could remain smooth. It is generally believed at present[23] (see also Ref. 24) that there is a weak, possibly essential, singularity of τ at T_R.

(ii) As already noted (last remark in Section 4) $b^{-1} \sim (T - T_c)^{-1/2}$ as $T \to T_c$. Assuming that (4.5) holds up to T_c (it seems hard to imagine at which lower temperature it would cease to be valid), we may rewrite σ^2 in (4.6) as a sum of two terms:

$$\sigma^2 = \langle D_0^2 \rangle + \sum_{i \neq 0} \langle D_0 D_i \rangle$$

If the second term $\sum_{i\neq 0}\langle D_0 D_i\rangle$ is positive or if it does not diverge any faster than $\langle D_0^2 \rangle$ as $T \to T_c$ (we are unable to prove either of these assumptions), then we conclude from (4.5) and (4.7) that the intrinsic width $\langle D_0^2 \rangle^{1/2}$, as defined in Section 3(e) diverges like $(T - T_c)^{-1/2}$ as $T \to T_c$, i.e., slower than the bulk correlation length $\xi \sim |T - T_c|^{-1}$.

If all the above assumptions were true then according to our definition, the intrinsic width would not diverge like the correlation length. A possible, very tentative, way out of this dilemma is to note that the deformations which are far from the vertical line $i_1 = 0$ have no influence on the shape of λ near this vertical line. If we compute a local magnetization profile along the line $i_1 = 0$ we can use instead of λ a phase separation line $\lambda_{l(T)}$ obtained by fixing the height of λ at the left-hand side of the interval $[-l(T), + l(T)]$ at zero. We get then a profile which depends on $l(T)$. We can now choose $l(T)$ to be of the order of the correlation length of the system. If (3.7) and (3.8) remain asymptotically correct for this interval $[-l(T), +l(T)]$ when $T \to T_c$ we should obtain together with (4.6) a width for the profile which diverges like the correlation length $\xi \sim |T - T_c|^{-1}$. We note that Weeks[8] has used a similar description of the intrinsic width of the continuum liquid–vapor interface in $d = 3$. He gives arguments that theories of van der Waals type, which predict a finite thickness at $T < T_c$, are indeed local theories of the interface, dealing essentially with a subsystem with an interfacial cross-sectional area of the order of the square of the correlation length.

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APPENDIX A.

We prove here Proposition 3.1 in a slightly more general form, namely, we show that the exponential decay of the (truncated) two-point correlation function implies that all the finite volume correlation functions approach their thermodynamic limit exponentially fast. This is based on correlation inequalities. Since the converse statement was proven by Martin–Löf,[17] we have therefore an equivalence, for Ising spin systems pair ferromagnetic interactions, between exponential approach to the thermodynamic limit and exponential decay of the correlation functions.

Note also that the exponential decay of the two-point function can be derived from the explicit solution[18] and the correlation inequalities of Ref. 19 for the state under consideration.

Proof of Proposition 3.1. Since $\sum_{i \in A} \sigma_i - \sigma_A$ is increasing in the F.K.G. sense[20] and since the state $\langle\ \rangle^+_{V_1 \cap V_2}$ dominates the state $\langle\ \rangle^+_{V_1}$ or $\langle\ \rangle^+_{V_2}$ in the same sense, we may write

$$\begin{aligned} |\langle \sigma_A \rangle^+_{V_1} - \langle \sigma_A \rangle^+_{V_2}| &\leqslant \langle \sigma_A \rangle^+_{V_1 \cap V_2} - \langle \sigma_A \rangle^+_{V_1} + \langle \sigma_A \rangle^+_{V_1 \cap V_2} - \langle \sigma_A \rangle^+_{V_2} \\ &\leqslant \sum_{i \in A} \langle \sigma_i \rangle^+_{V_1 \cap V_2} - \langle \sigma_i \rangle^+_{V_1} + \langle \sigma_i \rangle^+_{V_1 \cap V_2} - \langle \sigma_i \rangle^+_{V_2} \end{aligned}$$

But

$$\langle \sigma_i \rangle^+_{V_1 \cap V_2} - \langle \sigma_i \rangle^+_{V_1} = \int_0^\infty dh \left(\sum_{j \in V_1 \setminus V_2} \langle \sigma_i \sigma_j \rangle^h_{V_1} - \langle \sigma_i \rangle^h_{V_1} \langle \sigma_j \rangle^h_{V_1} \right) \quad \text{(A1)}$$

where $\langle\ \rangle^h_{V_1}$ is the state in V_1 with + boundary conditions and an external field h acting on all spins in $V_1 \backslash V_2$. By the G.H.S. inequality,[21] $\langle \sigma_i \sigma_j \rangle_V^h - \langle \sigma_i \rangle_{V_1}^h \langle \sigma_j \rangle_{V_2}^h$ is monotone decreasing in h and its value at $h = 0$ is increasing in V_1. Therefore

$$\langle \sigma_i \sigma_j \rangle_{V_1}^h - \langle \sigma_i \rangle_{V_1}^h \langle \sigma_j \rangle_{V_1}^h \leqslant \langle \sigma_i \sigma_j \rangle^+ - \langle \sigma_i \rangle^+ \langle \sigma_j \rangle^+ \leqslant \exp\left[-\bar{a}(i-j) \right] \tag{A2}$$

where the second inequality follows from the explicit solution.[18,19]

On the other hand we may estimate

$$\langle \sigma_i \sigma_j \rangle_{V_1}^h - \langle \sigma_i \rangle_{V_1}^h \langle \sigma_j \rangle_{V_1}^h \leqslant \exp(-ch) \tag{A3}$$

uniformly in V_1. This is because: Probability that $\sigma_i = +1 = \langle (1+\sigma_i)/2 \rangle_{V_1}^h \geqslant 1 - \exp(-ch)$ uniformly in V_1. To get the last inequality we bound $\langle (1+\sigma_i)/2 \rangle_{V_1}^h$ from below (Griffiths' inequalities[22]) by the expectation value of $(1+\sigma_i)/2$ taken at site i with no coupling with the outside which is equal to $\tanh h$.

Taking the product of the square roots of (A2) and (A3) in the r.h.s. of (A1) concludes the proof. ■

APPENDIX B.

We sketch the proof of the identification between the correlation functions of the deformations considered in this paper and those of Ref. 1. The problem is simply that in the sum (3.5) $P_L(\theta')$ is zero unless $D(\theta) = \sum_i D_i(\lambda(\theta)) = 0$, while in Ref. 1, Eq. (6.5), the definition of $P_L(\theta')$ given is extended by a formula like (3.4) to all admissible θ', also with $D(\theta') \neq 0$. This is because it is easier to use the Kirkwood–Salzburg equation for an ensemble of deformations which is not constrained by a long-range hard-core like $D(\theta) = 0$. Let $\rho_L(\theta)$ be defined as in (3.5). If we call $\varphi_L(\theta)$ the numerator of (3.4) and define it also for θ with $D(\theta) \neq 0$, we may write

$$\rho_L(\theta') = \frac{\sum_\theta \int_{-\pi}^{\pi} e^{itD(\theta+\theta')} \varphi_L(\theta+\theta')\, dt}{\sum_\theta \int_{-\pi}^{\pi} e^{itD(\theta)} \varphi_L(\theta)\, dt}$$

while $\rho_L'(\theta)$ which corresponds to Eq. (6.5) of Ref. 1 is

$$\rho_L'(\theta') = \frac{\sum_\theta \varphi_L(\theta'+\theta)}{\sum_\theta \varphi_L(\theta)}$$

We want to show the following.

Proposition. We have

$$\lim_{L\to\infty} \frac{\rho_L(\theta')}{\rho_L'(\theta')} = 1 \qquad \text{for all } \theta'$$

Proof. For this it is enough to show that

$$\frac{1/2\pi \int_{-\pi}^{\pi} \sum_{\theta} \varphi_L(\theta) e^{itD(\theta)}\, dt}{\sum_{\theta} \varphi_L(\theta)} = \frac{c}{\sqrt{L}} + o\left(\frac{1}{\sqrt{L}}\right) \tag{B1}$$

and

$$\frac{1/2\pi \int_{-\pi}^{\pi} \sum_{\theta} \varphi_L(\theta + \theta') e^{itD(\theta+\theta')}\, dt}{\sum_{\theta} \varphi_L(\theta + \theta')} = \frac{c}{\sqrt{L}} + o\left(\frac{1}{\sqrt{L}}\right) \tag{B2}$$

with the same c in both equations $[c = [(2\pi)^{1/2}\sigma]^{-1}]$. The first equation is exactly (8.13) of Ref. 1.

To prove the second equation, define $\varphi'_L(\theta) = \varphi_L(\theta + \theta')/\varphi_L(\theta')$.

It is not hard (but lengthy) to show that using the technique of Ref. 1 one can derive the same Kirkwood–Salzburg equations for the Gibbs factor $\varphi'_L(\theta)$ as for the $\varphi_L(\theta)$ and that the corresponding φ^T satisfy

$$\sum_{\theta} \left|\varphi_L'^T(\theta) - \varphi_L^T(\theta)\right| < \infty \tag{B3}$$

uniformly in L. This last estimate is because the influence of θ' is exponentially small on the $\varphi^T(\theta)$ for θ far from θ'.

Then we write the right-hand side of (B2) as

$$\frac{1/2\pi \int_{-\pi}^{\pi} \sum_{\theta} \varphi'_L(\theta) e^{itD(\theta)} e^{itD(\theta')}\, dt}{\sum_{\theta} \varphi'_L(\theta)}$$

Applying the same arguments (with φ'_L instead of φ_L) that lead to (8.13) in Ref. 1 with $k = -D(\theta')$ we obtain the same result but with a σ^2 defined via Ref. 1, Eq. (8.8), with φ' instead of φ. However, owing to (B3) the two σ^2 coincide. Moreover since $k = D(\theta')$ is independent of N, $\exp(-k/2N\sigma^2) = 1 + o(1/N)$ which finishes the proof.

REFERENCES

1. G. Gallavotti, *Commun. Math. Phys.* **27**:103 (1972).
2. D. B. Abraham and P. Reed, *Commun. Math. Phys.* **43**:35 (1976).
3. F. P. Buff, R. A. Lovett, and F. H. Stillinger, *Phys. Rev. Lett.* **15**:621 (1965).
4. N. J. Günther, D. A. Nicole, and D. J. Wallace, *J. Phys. A.* **13**:1755 (1980).
5. J. P. van der Waals, *Z. Phys. Chem.* **13**:657 (1894); translation in *J. Stat. Phys.* **20**:137 (1979).
6. J. W. Cahn and J. E. Hilliard, *J. Chem. Phys.* **28**:258 (1958).

7. B. Widom, *Structure and Thermodynamics of Interfaces in Statistical Mechanics and Statistical Methods in Theory and Application.* V. Lanzman, ed. (Plenum, New York, 1977).
8. J. D. Weeks, *J. Chem. Phys.* **67**:3106 (1977).
9. J. S. Rowlinson, *Adv. Chem. Phys.* **41**:1, 1980.
10. G. Gallavotti, *Riv. Nuovo Cimento* **2**:133 (1972).
11. R. L Dobrushin, *Theory Prob. Appl.* **17**:582 (1972); *Theory Prob. Appl.* **18**:253 (1973).
12. H. van Beijeren, *Commun. Math. Phys.* **40**:1 (1975).
13. G. Gallavotti, A. Martin-Löf, and S. Miracle-Sole, in *Statistical Mechanics and Mathematical Problems, Battelle Recontres 1971, Lecture Notes in Physics*, Vol. 20, (Springer, Berlin, 1973).
14. J. Bricmont, J. L. Lebowitz, and C. E. Pfister, *Commun. Math. Phys.* **66**:21 (1979).
15. D. B. Abraham, G. Gallavotti, and A. Martin-Löf, *Physica (Utrecht)* **65**:73 (1973).
16. J. D. Weeks, G. H. Gilmer, and H. J. Leamy, *Phys. Rev. Lett.* **31**:549 (1973).
17. A. Martin-Löf, *Commun. Math. Phys.* **32**:75 (1973).
18. B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Harvard University Press, Cambridge, Massachusetts. 1973).
19. J. L. Lebowitz, *J. Stat. Phys.* **16**:463 (1977).
20. C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre, *Commun. Math. Phys.* **22**:89 (1971).
21. R. B. Griffiths, C. A. Hurst, and S. Sherman, *J. Math Phys.* **11**:790 (1970).
22. R. B. Griffiths, *J. Math. Phys.* **8**:478 (1967).
23. G. Itzykson, M. E. Peskin, and J. B. Zuber, *Phys. Lett.* **95B**:259 (1980) and references quoted therein.
24. B. Durhuus and J. Fröhlich, *Commun. Math. Phys.* **75**:103 (1980).